

Large joints in graphs

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January 13, 2010

Abstract

We show that if $r \geq s \geq 2$, $n > r^8$, and G is a graph of order n containing as many r -cliques as the r -partite Turán graph of order n , then G has more than at $n^{r-1}/(4r)^{r+6}$ cliques sharing a common edge unless G is isomorphic to the the r -partite Turán graph of order n . This structural result generalizes a previous result that has been useful in extremal combinatorics.

Keywords: *joint; jointsize; clique; number of cliques; Turán graph*

Introduction

In notation we follow [3]; in particular, $T_r(n)$ denotes the r -partite Turán graph of order n and $t_r(n)$ denotes the number of its edges. Also, an r -joint of size t is a collection of t distinct r -cliques sharing an edge. (Note that two r -cliques of an r -joint may share $r-1$ vertices.) We write $\text{js}_r(G)$ for the maximum size of an r -joint in a graph G ; in particular, if $2 \leq r \leq n$ and r divides n then $\text{js}_r(K_n) = \binom{n-2}{r-2}$ and $\text{js}_r(T_r(n)) = \left(\frac{n}{r}\right)^{r-2}$.

In [5] we improved a result of Erdős [8] to the following assertion.

Let $r \geq 2$, $n > r^8$, and let G be a graph of order n and size at least $t_r(n)$. Then

$$\text{js}_{r+1}(G) > \frac{n^{r-1}}{r^{r+5}} \quad (1)$$

unless $G = T_r(n)$.

Joints have a long history in graph theory. The study of $\text{js}_3(G)$, also known as the *booksize* of G , was initiated by Erdős in [6] and subsequently generalized in [7] and [8]; it seems that he foresaw the importance of joints when he restated his general results in 1995, in [9]. A quintessential result concerning joints is the “triangle removal lemma” of Ruzsa and Szemerédi [16], which can be stated

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[‡]Research supported in part by NSF grants DMS-0505550, CNS-0721983 and CCF-0728928, and ARO grant W911NF-06-1-0076

[§]Research supported by NSF Grant # DMS-0906634

as a lower bound on $\text{js}_3(G)$ when G is a graph of a particular kind. Erdős's challenge was taken up in Ramsey graph theory, see, e.g., [15] and its references. Some recent applications are given in [13, 14].

Our aim in this note is to prove an analogue of inequality (1) in the case when G has a fair number of r -cliques, rather than edges. More precisely, letting $k_r(G)$ stand for the number of r -cliques of a graph G , we shall prove the following theorem.

Theorem 1 *Let $r \geq s \geq 2$, $n > r^8$, and let G be a graph of order n , with $k_s(G) \geq k_s(T_r(n))$. Then*

$$\text{js}_{r+1}(G) > \frac{n^{r-1}}{(4r)^{r+6}} \quad (2)$$

unless $G = T_r(n)$.

Inequality (2) is far from the best possible; in particular, for $s = 2$ inequality (1) is significantly better. However, in most applications, the exact values of the coefficients to n^{r-1} in (1) and (2) are irrelevant, except for the convenience. Moreover, these inequalities cannot be improved too much, as shown by the graph G obtained by adding an edge to $T_r(n)$: if n is a multiple of r then $k_s(G) \geq k_s(T_r(n))$ and $\text{js}_{r+1}(G) = \left(\frac{n}{r}\right)^{r-1}$.

We very much hope that Theorem 1 will be one of many new generalizations of classical extremal results in graph theory to be proved in the near future.

Preliminary results

In this section we shall collect the results we shall use in our proof of Theorem 1. The first two, stated as 'Facts', are from earlier papers, but the two lemmas following them seem to be new. We shall also need two simple inequalities about the Turán graph $T_r(n)$. The required proofs of the results below will be given in the next section.

We start with an inequality stated by Moon and Moser in [12]; it seems that Khadžiivanov and Nikiforov [10] were the first to publish a complete proof of this (see also [11], Problem 11.8).

Fact 2 *Let $1 \leq s < t < n$, and let G be a graph of order n containing at least one t -clique. Then*

$$\frac{(t+1)k_{t+1}(G)}{tk_t(G)} - \frac{n}{t} \geq \frac{(s+1)k_{s+1}(G)}{sk_s(G)} - \frac{n}{s}. \quad (3)$$

The second fact we need is a stability theorem, stated as Theorem 9 in [5].

Fact 3 *Let*

$$r \geq 2, \quad n > r^8 \text{ and } 0 < \beta < r^{-8}/16;$$

furthermore, let G be a graph of order n and size

$$e(G) > \left(\frac{r-1}{2r} - \beta \right) n^2.$$

Then either

$$\text{js}_{r+1}(G) > \frac{n^{r-1}}{r^{r+6}}, \quad (4)$$

or G contains an induced r -partite subgraph G_0 of order at least $(1 - 2\sqrt{\beta})n$ with minimum degree

$$\delta(G_0) > \left(1 - \frac{1}{r} - 4\sqrt{\beta}\right)n. \quad (5)$$

Let us turn to the two technical lemmas, which seem to be new. The first one is somewhat paradoxical: informally it says that if a graph G contains few $(r+1)$ -cliques, then the ratio $k_2(G)/k_r(G)$ is large.

Lemma 4 *Let $\alpha \geq 0$ and G be a graph of order n . If*

$$k_{r+1}(G) < \frac{\alpha r^2}{r+1} \left(\frac{n}{r}\right)^{r+1},$$

then

$$k_2(G) > \frac{rk_r(G)}{2n^{r-2}} \prod_{s=2}^{r-1} \left(\frac{r-s}{rs} + \alpha\right)^{-1}.$$

Lemma 5 *Let $\alpha > 0$, $2 \leq s \leq r \leq n$, and let G be a graph of order n . If $k_r(G) \geq k_r(T_r(n))$, then either*

$$\text{js}_{r+1}(G) > \alpha r \left(\frac{n}{r}\right)^{r-1}$$

or

$$k_2(G) > \left(\frac{r-1}{2r} - \frac{r^3\alpha}{2} - \frac{r^3}{16n^2}\right)n^2.$$

Finally, the following two inequalities about Turán graphs are easily checked.

Fact 6 *For every $2 \leq r \leq n$,*

$$k_2(T_r(n)) \geq \frac{r-1}{2r}n^2 - \frac{r}{8}. \quad (6)$$

$$k_r(T_r(n)) \geq \left(\frac{n}{r}\right)^r - \frac{r^2}{16} \left(\frac{n}{r}\right)^{r-2}. \quad (7)$$

Proofs

In this section we shall prove Lemmas 4 and 5, and Theorem 1.

Proof of Lemma 4. We have

$$\frac{(r+1)k_{r+1}(G)}{rk_r(G)} < \alpha r \left(\frac{n}{r}\right)^{r+1} \left(\frac{n}{r}\right)^{-1} \leq \alpha r \left(\frac{n}{r}\right)^{r+1} \left(\frac{n}{r}\right)^{-r} = \alpha n.$$

Now, for every $s = 2, \dots, r-1$, inequality (3) gives

$$\frac{(s+1)k_{s+1}(G)}{sk_s(G)} - \frac{n}{s} \leq \frac{(r+1)k_{r+1}(G)}{rk_r(G)} - \frac{n}{r} \leq \alpha n - \frac{n}{r},$$

and so,

$$\frac{(s+1)k_{s+1}(G)}{sk_s(G)} \leq \left(\frac{r-s}{sr} + \alpha \right) n.$$

Multiplying these inequalities for $s = 2, \dots, r-1$, we obtain

$$2k_2(G) n^{r-2} \prod_{s=2}^{r-1} \left(\frac{r-s}{rs} + \alpha \right) \geq rk_r(G),$$

and the desired inequality follows. □

Proof of Lemma 5. Assume that $js_{r+1}(G) \leq \alpha r (n/r)^{r-1}$. Then

$$\binom{r+1}{2} k_{r+1}(G) \leq js_{r+1}(G) k_2(G) < \alpha r \left(\frac{n}{r} \right)^{r-1} \frac{n^2}{2}$$

and so,

$$k_{r+1}(G) \leq \alpha \frac{r^2}{r+1} \left(\frac{n}{r} \right)^{r+1}.$$

Now Lemma 4 and inequality (7) give

$$\begin{aligned} k_2(G) &> \frac{rk_r(G)}{2n^{r-2}} \prod_{i=2}^{r-1} \left(\frac{r-i}{ri} + \alpha \right)^{-1} \\ &> r \left(\frac{1}{r} \right)^{r-2} \left(\left(\frac{n}{r} \right)^2 - \frac{r^2}{16} \right) \prod_{i=2}^{r-1} \left(\frac{r-i}{ri} + \alpha \right)^{-1}. \end{aligned}$$

Furthermore, note that

$$\begin{aligned} \prod_{i=2}^{r-1} \left(\frac{r-i}{ri} + \alpha \right) &= \prod_{i=2}^{r-1} \left(1 + \frac{ri}{r-i} \alpha \right) \prod_{i=2}^{r-1} \left(\frac{r-i}{ri} \right) \leq (1 + r(r-1)\alpha)^{r-2} \prod_{i=2}^{r-1} \left(\frac{r-i}{ri} \right) \\ &= (1 + r(r-1)\alpha)^{r-2} \left(\frac{1}{r} \right)^{r-2} \frac{r-2}{2} \cdot \frac{r-3}{3} \cdot \dots \cdot \frac{2}{r-2} \cdot \frac{1}{r-1} \\ &= \left(\frac{1}{r} \right)^{r-2} \frac{1}{r-1} (1 + r(r-1)\alpha)^{r-2}. \end{aligned}$$

Hence, by (6), we see that

$$\begin{aligned} k_2(G) &> \binom{r}{2} \frac{1}{(1 + r(r-1)\alpha)^{r-2}} \left(\left(\frac{n}{r} \right)^2 - \frac{r^2}{16} \right) \\ &> \left(\frac{r-1}{2r} \right) (1 - r^2\alpha)^{r-2} \left(1 - \frac{r^4}{16n^2} \right) n^2 \\ &> \left(\frac{r-1}{2r} - \frac{r^3\alpha}{2} - \frac{r^4}{16n^2} \right) n^2. \end{aligned}$$

as claimed. \square

After all this preparation, we are ready to prove our main result.

Proof of Theorem 1. As shown in [1] (see also [2], p.??), if $k_s(G) \geq k_s(T_r(n))$, then $k_r(G) \geq k_r(T_r(n))$. Consequently, we may assume that $s = r$. Also, assume for a contradiction that

$$js_{r+1}(G) \leq \frac{n^{r-1}}{(4r)^{r+6}}. \quad (8)$$

First, setting $\alpha = 4^{-r-6}r^{-7}$, Lemma 5 implies that

$$e(G) > \left(\frac{r-1}{2r} - \frac{r^3\alpha}{2} - \frac{r^4}{16n^2} \right) n^2 > \left(\frac{r-1}{2r} - \frac{1}{4r^{12}} \right) n^2.$$

Now, recalling that $n > r^8$ and setting $\beta = r^{-12}/4$, by Fact 3 we find that G contains an induced r -partite subgraph G_0 with $|G_0| \geq (1 - r^{-6})n$ and minimum degree $\delta(G_0) > (1 - 1/r - 2r^{-6})n$.

Let V_1, \dots, V_r be the vertex classes of G_0 , set $V_0 = V(G) \setminus V(G_0)$, and let U be the set of vertices in V_0 joined to a vertex of each V_1, \dots, V_r . Set for short $\varepsilon = 2r^{-6}$ and $\delta = \delta(G_0)$. It turns out that none of the vertex classes is significantly larger than n/r . Indeed, for every $i \in [r]$, we see that

$$|V_i| \leq |G_0| - \delta \leq n - (1 - 1/r - \varepsilon)n = \left(\frac{1}{r} + \varepsilon \right) n. \quad (9)$$

Before giving further details, we shall outline the remaining steps of our proof in three formal claims.

Claim 1. *For every $u \in U$, there exist two distinct elements $i, j \in [r]$ such that*

$$|\Gamma(u) \cap V_i| < \frac{n}{3r} \quad \text{and} \quad |\Gamma(u) \cap V_j| < \frac{n}{3r}.$$

Claim 2. *Every vertex $u \in U$ belongs to at most $0.91(n/r)^{r-1}$ distinct r -cliques of G .*

Claim 3. *If U is non-empty, then $k_r(G) < k_r(T_r(n))$.*

The last Claim gives us a contradiction if $U \neq \emptyset$. However, if U is empty, the graph G is r -partite and $k_r(G) \leq k_r(T_r(n))$, with equality if and only if $G = T_r(n)$. Hence, to complete our proof of Theorem 1, all that remains is to prove these claims.

Proof of Claim 1. Assume for a contradiction that there is $u \in U$ such that

$$|V_i \cap \Gamma(u)| \geq \frac{1}{3r}n$$

for all but at most one $i \in [r]$; if there is such an i , we may assume that $i = 1$. Choose $v_1 \in V_1 \cap \Gamma(u)$; we shall prove that the edge uv_1 is contained in at least $(n/4r)^{r-1}$ distinct $(r+1)$ -cliques. This will give $js_{r+1}(G) \geq (n/4r)^{r-1}$, contradicting the assumption (8).

Let $2 \leq s \leq r-1$ and choose any $s-1$ vertices $v_i \in V_i$, $i \in [2..r]$. Letting

$$b = |V_{s+1} \cap \Gamma(u) \cap (\cap_{i=1}^s \Gamma(v_i))|,$$

we shall prove that $b > n/(4r)$. Indeed, for every $i \in [2..s]$, note that

$$\begin{aligned} |V_{s+1} \cap \Gamma(v_i)| &= |V_{s+1}| + |\Gamma(v_i)| - |V_{s+1} \cup \Gamma(v_i)| \geq |V_{s+1}| + \delta - (n' - |V_i|) \\ &= |V_{s+1}| + \delta - n + |V_i|. \end{aligned}$$

Now, we find that

$$\begin{aligned} b &= |V_{s+1} \cap \Gamma(u) \cap (\cap_{i=1}^s \Gamma(v_i))| \\ &\geq |V_{s+1} \cap \Gamma(u)| + |\cap_{i=1}^s (V_{s+1} \cap \Gamma(v_i))| - |V_{s+1}| \\ &\geq \frac{1}{3r}n + \left(\sum_{i=2}^s |V_{s+1} \cap \Gamma(v_i)| - (s-1)|V_{s+1}| \right) - |V_{s+1}| \\ &\geq \frac{1}{3r}n + \sum_{i=2}^s (|V_{s+1}| + \delta - n + |V_i|) - s|V_{s+1}| > \frac{1}{2r}n + \sum_{i=2}^s (\delta + |V_i| - n) \\ &> \frac{1}{3r}n + \sum_{i=1}^r (\delta + |V_i| - n) = \frac{1}{2r}n + r\delta + n' - rn \\ &> \frac{1}{3r}n + (r-1-r\varepsilon)n + (1-\varepsilon)n - rn > \left(\frac{1}{3r} - (r+1)\varepsilon \right)n \\ &> \frac{1}{4r}n. \end{aligned}$$

To bound the number of cliques containing uv_1 , for $s = 2, \dots, r$, choose a vertex v_s such that

$$v_s \in V_s \cap \Gamma(u) \cap (\cap_{i=1}^{s-1} \Gamma(v_i)).$$

Clearly for every choice of v_2, \dots, v_r , the set $\{u, v_1, v_2, \dots, v_r\}$ induces an $(r+1)$ -clique. Since for every $s = 2, \dots, r$, the vertex v_s can be chosen in at least $n/(4r)$ ways, there are at least $(n/4r)^{r-1}$ distinct $(r+1)$ -cliques containing the edge uv_1 , completing the proof of Claim 1. \square

Proof of Claim 2. Fix a vertex $u \in U$ and let K be the set of all r -cliques containing u . By Claim 1, we can assume that

$$|\Gamma(u) \cap V_1| < \frac{n}{3r} \quad \text{and} \quad |\Gamma(u) \cap V_2| < \frac{n}{3r}.$$

Write K_s for the set of $(r-1)$ -cliques in K intersecting $V(G_0)$ in exactly s vertices and note that

$$|K| = |K_0| + |K_1| + \dots + |K_{r-1}|.$$

Since G_0 is r -partite and each vertex class satisfies (9), for every $s = 1, \dots, r-1$,

$$k_s(G_0) \leq \binom{r}{s} \left(\frac{1}{r} + \varepsilon \right)^s n^s.$$

On the other hand, for $s = 1, \dots, r-1$ there are at most $\binom{\varepsilon n}{s}$ s -cliques entirely outside G_0 . Thus, for every $s = 1, \dots, r-2$, we have

$$|K_s| < \binom{\varepsilon n}{r-1-s} \binom{r}{s} \left(\frac{1}{r} + \varepsilon \right)^s n^s.$$

It is easy to check that the right-hand side of this inequality increases with s , and so

$$\begin{aligned}
|K_1| + \cdots + |K_{r-1}| &< (r-1) \varepsilon n \binom{r}{r-2} \left(\frac{1}{r} + \varepsilon\right)^{r-2} n^{r-2} \\
&< \frac{r^3}{2} \varepsilon \left(\frac{1}{r} + \varepsilon\right)^{r-2} n^{r-1} < \frac{1}{r^3} \left(\frac{1}{r} + \varepsilon\right)^{r-2} n^{r-1} \\
&< \frac{1}{r^2} \left(\frac{1}{r} + \varepsilon\right)^{r-1} n^{r-1} \leq \frac{1}{9} \left(\frac{1}{r} + \varepsilon\right)^{r-1} n^{r-1}.
\end{aligned} \tag{10}$$

Looking closely at K_0 , it turns out that K_0 is the union of the following three disjoint sets:

$$\begin{aligned}
K'_0 &= \{R : R \in K_0, R \cap V_1 \neq \emptyset, R \cap V_2 = \emptyset\}, \\
K''_0 &= \{R : R \in K_0, R \cap V_2 \neq \emptyset, R \cap V_1 = \emptyset\}, \\
K'''_0 &= \{R : R \in K_0, R \cap V_1 \neq \emptyset, R \cap V_2 \neq \emptyset\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
|K_0| &= |K'_0| + |K''_0| + |K'''_0| \leq 2 \frac{1}{3r} n \left(\frac{1}{r} + \varepsilon\right)^{r-3} n^{r-2} + \frac{1}{9r^2} n^2 \left(\frac{1}{r} + \varepsilon\right)^{r-3} n^{r-3} \\
&= \frac{2}{3} \left(\frac{1}{r} + \varepsilon\right)^{r-1} n^{r-1} + \frac{1}{9} \left(\frac{1}{r} + \varepsilon\right)^{r-1} n^{r-1} = \frac{7}{9} \left(\frac{1}{r} + \varepsilon\right)^{r-1} n^{r-1}.
\end{aligned}$$

Hence, in view of (10),

$$|K| < \frac{8}{9} \left(\frac{1}{r} + \frac{2}{r^6}\right)^{r-1} n^{r-1} < \frac{8}{9} \left(1 + \frac{2}{r^5}\right)^{r-1} \left(\frac{n}{r}\right)^{r-1} < 0.91 \left(\frac{n}{r}\right)^{r-1},$$

completing the proof of Claim 2. □

Proof of Claim 3. First note that

$$\begin{aligned}
k_r(T_r(n)) - k_r(T_r(n-1)) &\geq \left(\frac{n}{r} - 1\right)^{r-1} > \left(\frac{n}{r}\right)^{r-1} \left(1 - \frac{r(r-1)}{n}\right) \\
&> \left(1 - \frac{r(r-1)}{r^8}\right) \left(\frac{n}{r}\right)^{r-1} \\
&> \left(\frac{n}{r}\right)^{r-1} \left(1 - \frac{2}{3^7}\right) > 0.99 \left(\frac{n}{r}\right)^{r-1}.
\end{aligned}$$

In particular, this implies that

$$k_r(T_r(n)) - k_r(T_r(n - |U|)) > 0.99 |U| \left(\frac{n - |U|}{r}\right)^{r-1}.$$

According to Claim 2, by removing the set U we destroy at most

$$0.91 |U| \left(\frac{n}{r}\right)^{r-1}$$

r -cliques. But the graph induced by $V(G) \setminus U$ is r -partite and so, according to Zykov's theorem, [17], it has at most $k_r(T_r(n - |U|))$ r -cliques. Thus,

$$k_r(T_r(n - |U|)) + 0.91|U| \left(\frac{n}{r}\right)^{r-1} \geq k_r(G) \geq k_r(T_r(n)),$$

implying in turn that

$$0.91 \left(\frac{n}{r}\right)^{r-1} > 0.99 \left(\frac{n - |U|}{r}\right)^{r-1},$$

and so

$$\frac{0.91}{0.99} > \left(1 - \frac{|U|}{n}\right)^{r-1} > \left(1 - \frac{1}{r^6}\right)^{r-1} > \left(1 - \frac{1}{3^6}\right)^2.$$

This contradiction completes the proof of Claim 3 and Theorem 1. \square

It would be good to determine the best constant in Theorem 2, the maximal c such that if $2 \leq s \leq r$ are fixed, $n \rightarrow \infty$, and G is a graph of order n with $k_s(G) \geq k_s(T_r(n))$ then $\text{js}_{r+1}(G) \geq (c + o(1))n^{r-1}$ unless $G = T_r(n)$. For $s = r = 2$, it is known that the best constant is $1/6$ see [4] and the references therein. For larger values of r , we do not expect this task to be easy.

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